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Generalized Goldberg–Sachs theorems for pseudo-Riemannian four-manifolds

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Abstract

It has been recently observed that the generalized Goldberg–Sachs theorem in general relativity as well as some of its corollaries admit appropriate Riemannian versions. In this paper we use the formalism of spinors to give alternative proofs of these results clarifying the analogy between positive Hermitian structures of oriented Riemannian four-manifolds and shear-free congruences of oriented Lorentzian four-manifolds. We also prove similar results for oriented pseudo-Riemannian four-manifolds when the metric is of zero signature. This allows us to describe compact oriented four-manifolds possibly admitting a pseudo-Riemannian Einstein metric of zero signature whose positive Weyl tensor has two distinct eigenvalues corresponding to non-isotropic eigenspaces.

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1. Introduction

The *Goldberg–Sachs theorem* in general relativity treats the geometry of the *shear-free congruences* of an oriented Lorentzian four-manifold satisfying the vacuum field equation. It was observed in [13,16] that this theorem admits a formulation in a Riemannian framework, which says that an oriented Riemannian Einstein four-manifold M admits (locally) a positive orthogonal complex structure if and only if the spectrum of the positive Weyl tensor

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(considered as an endomorphism of the bundle Λ^+M of self-dual 2-forms) is degenerate. This result was successfully used by Boyer [3] and LeBrun [12] to study compact Einstein Hermitian surfaces.

A detailed analysis of the local existence of positive orthogonal complex structures on oriented Riemannian four-manifolds was given in [2] where some other applications to the geometry of Hermitian surfaces where obtained. Summarizing Theorem 1 and Proposition 3 in [2] we obtain the following Riemannian version of the *Generalized Goldberg–Sachs Theorem* ([9,14, Proposition 7.3.35, 21]).

Theorem A. *Let (M, g) be an oriented Riemannian four-manifold with nowhere vanishing positive Weyl tensor W^+. Suppose that J is a positive g-orthogonal principal almost-complex structure on M, i.e. W^+ has no component in the bundle $Hom(\Lambda_0^+M)$, where Λ_0^+M denotes the vector bundle of J-anti-invariant 2-forms on (M, J). Then any two of the following three conditions imply the third:*

(i) *the spectrum of W^+ is degenerate;*
(ii) *J is integrable;*
(iii) *the codifferential δW^+ of W^+ vanishes on any triple of $(1, 0)$-vector fields.*

Since any positive g-orthogonal integrable almost-complex structure is principal (see [22]), we obtain using the second Bianchi identity and Theorem A that the spectrum of the positive Weyl tensor of a Hermitian surface with J-invariant Ricci tensor is degenerate [2, Theorem 2], a result which corresponds to the Robinson–Shild Theorem in the Lorentzian case ([17], see also [14, Proposition 7.3.43]).

The complete description of the irreducible components of the covariant derivative DW^+ of W^+ under the action of the unitary group $U(2)$ given in [2] leads to similar results by considering the *Penrose operator* P instead of the codifferential δ, i.e. the complementary part PW^+ of δW^+ in the covariant derivative DW^+, (see [2, Propositions 5 and 6]). We have the following:

Theorem B. *Let (M, g) be an oriented Riemannian four-manifold. Suppose that J is a positive g-orthogonal principal almost-complex structure on M. Then*

(i) *if the spectrum of W^+ is everywhere non-degenerate, J is integrable if and only if $(PW^+)(Z_1, Z_2, Z_3, Z_4, Z_5) = 0$ for any $(1, 0)$-vector fields Z_1, Z_2, Z_3, Z_4, Z_5;*
(ii) *if the spectrum of W^+ is everywhere degenerate, but W^+ nowhere vanishes, J is integrable if and only if $(PW^+)(Z_1, Z_2, Z_3, Z_4, X) = 0$ for any $(1, 0)$-vector fields Z_1, Z_2, Z_3, Z_4 and any vector field X;*
(iii) *if J is integrable, the spectrum of W^+ is degenerate if and only if $(PW^+)_{Z_1,Z_2,Z_3,Z_4,X} = 0$ for any $(1, 0)$-vector fields Z_1, Z_2, Z_3, Z_4 and any vector field X.*

It follows from Theorem B that if (M, g) is an oriented Riemannian four-manifold for which $PW^+ = 0$, then the spectrum of W^+ is everywhere degenerate. Moreover, the eigenspace corresponding to the simple eigenvalue of W^+ determines a Kähler structure on the open set where W^+ is non-zero, up to a two-fold covering and a conformal change of the metric [2, Theorem 3]. In this note we use the formalism of spinors to give alternative proofs of Theorem A and Theorem B. These theorems can be rewritten in a "spinorial" form (see Theorem 1 and Theorem 2 in Section 4) which allows one to provide analogous results for arbitrary pseudo-Riemannian four-manifolds. In such a way we make transparent the analogy between positive Hermitian structures of oriented Riemannian four-manifolds and shear-free congruences of oriented Lorentzian four-manifolds. In particular, we provide the following Lorentzian version of the above mentioned Theorem 3 in [2]:

Theorem C. *Let (M, g) be an oriented Lorentzian four-manifold. Assume that the Weyl tensor W is annihilated by the Penrose operator. Then any principal null direction of W is multiple, i.e. W belongs (at any point of M) to one of the Petrov classes D, N or O.*

The proof of Theorem A presented here follows that given in [21] (see also [14, Proposition 7.3.35]), while Theorem B and Theorem C seem to have not appeared in the literature.

Finally, in Section 5, we discuss briefly oriented pseudo-Riemannian four-manifolds (M, g) with metric g of signature 0. In this case one can see that Theorem A and Theorem B (as well as the corollaries mentioned above) hold true by considering the *negative* orthogonal almost-complex structures of (M, g). As an application of Theorem A we obtain the following result closely related to [6] and [15]:

Theorem D. *Let (M, g) be an oriented pseudo-Riemannian Einstein four-manifold with a metric of signature 0. Suppose that at any point of M the positive Weyl tensor W^+ has exactly two distinct eigenvalues corresponding to non-isotropic eigenspaces. Then, replacing M by a two-fold covering if necessary, (M, g) admits a negative g-orthogonal complex structure J. Moreover, if M is compact, then (M, J) is either a ruled surface or a minimal surface of class VII_0 with no global spherical shell, and with second Betti number even and positive.*

Conversely, it follows from Theorem A (see the proof of [2, Proposition 1]) that the positive Weyl tensor of an indefinite Hermitian Einstein metric g on a complex surface (M, J) either vanishes identically or satisfies the conditions of Theorem D. The only known examples of such metrics are the natural indefinite Kähler–Einstein metrics on the minimal irrational ruled surfaces which are the total space of a flat $\mathbf{CP}^1$-bundle over a curve S of genus $g \geq 2$ (see [15]). It is still an open problem whether the other surfaces described in Theorem D do admit indefinite Hermitian Einstein metrics.

2. Positive orthogonal almost-complex structures on oriented Riemannian four-manifolds

In this section we will use the spinorial formalism to describe the properties of the positive g-orthogonal almost-complex structures of an oriented Riemannian four-manifold (M, g). We refer to [18,19] for more details and proofs.

Starting from the fact that the simply-connected double-covering group Spin (4) of $SO(4)$ splits as

$$\mathrm{Spin}(4) = Sp(1) \times Sp(1)$$

we will consider the rank 2 complex vector bundles Σ_+M and Σ_-M of positive and negative spinors associated to an oriented Riemannian *spin* four-manifold (M, g). (These bundles are locally defined on any Riemannian four-manifold since locally a spin structure always exists.) Thus Σ_+M and Σ_-M are naturally equipped with a quaternionic and a (real) symplectic structure and we have the following decomposition:

$$\Sigma_+^p M \otimes \Sigma_+^q M = \Sigma_+^{p+q} M \oplus \Sigma_+^{p+q-2} M \oplus \cdots \oplus \Sigma_+^{p-q} M, \quad p \geq q, \tag{1}$$

where $\Sigma_+^k M$ denotes the kth symmetric tensor power of Σ_+M. (The same formula holds for Σ_-M).

Denote by j_+ and ϵ_+ (resp. j_- and ϵ_-) the quaternionic conjugation and the symplectic form of Σ_+M (resp. Σ_-M). Then we may identify the complexified tangent bundle $TM \otimes \mathbf{C}$ of M with $\Sigma_+M \otimes \Sigma_-M$. (Here and henceforth we will freely identify Σ_+M and Σ_-M with the corresponding dual bundles via the symplectic structures ϵ_+ and ϵ_-). Thus the complex conjugation in $TM \otimes \mathbf{C}$ is induced by the operator $\sigma = j_+ \times j_-$ and the $\mathbf{C}$-linear symmetric extension of g on $TM \otimes \mathbf{C}$ can be expressed in terms of the symplectic forms ϵ_+ and ϵ_- in the following way:

$$g(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) = \epsilon_+(\xi_1, \xi_2)\epsilon_-(\eta_1, \eta_2), \quad \forall \xi_1, \xi_2 \in \Sigma_+M, \quad \forall \eta_1, \eta_2 \in \Sigma_-M. \tag{2}$$

We also identify the vector bundle $\Lambda^2 M \otimes \mathbf{C}$ of complex 2-forms of M with $\Sigma_+^2 M \oplus \Sigma_-^2 M$. Then the subbundle $\Lambda^+ M$ of self-dual 2-forms of M is isomorphic to $[\Sigma_+^2 M]$, where [ ] indicates the inverse of the complexification. Moreover, the bundle $\mathrm{Sym}_0(\Lambda_+^2 M)$ of traceless symmetric endomorphisms of $\Lambda_+^2 M$ can be identified with $[\Sigma_+^4 M]$.

For any point $x \in M$ the set of positive orthogonal almost-complex structures on (TM_x, g) is parametrized by the 2-sphere $\mathbf{P}(\Sigma_+M_x)$ since any such structure J is uniquely determined by its $(1, 0)$-space, i.e. by a g-isotropic complex 2-plane of $TM_x \otimes \mathbf{C}$ and any such a plane has the form $\{\xi \otimes \eta : \eta \in \Sigma_-M_x\}$, where ξ is a non-vanishing element of Σ_+M_p, uniquely determined up to multiplication by a non-zero complex scalar. For any $[\xi] \in \mathbf{P}(\Sigma_+M)$ we denote by $T_\xi M$ the $(1, 0)$-space of the positive orthogonal almost-complex structure J_ξ. Note that the quaternionic conjugate $\bar{\xi} = j_+(\xi)$ of ξ corresponds to the almost-complex structure $-J_\xi$.

If we denote by $F_\xi(.,.) = g(J_\xi.,.)$ the Kähler form of J_ξ and by $\Lambda_\xi^{2,0} M$ and $\Lambda_\xi^{0,2} M$ respectively the bundles of (2, 0) and (0, 2) 2-forms we have (see [19])

$$\mathbf{R}F_\xi = -i\mathbf{R}\xi \odot \bar{\xi}; \quad \Lambda_\xi^{2,0} M = \mathbf{C}(\xi \otimes \xi); \quad \Lambda_\xi^{0,2} = \mathbf{C}(\bar{\xi} \otimes \bar{\xi}),$$

where $\odot$ denotes the symmetric product.

We will say that a section $[\xi]$ of $\mathbf{P}(\Sigma_+M)$ is *integrable* if the corresponding almost-complex structure J_ξ is integrable. The Levi–Civita connection on (M, g) induces on any of the bundles $\Sigma_+ M$ and $\Sigma_- M$ a linear connection $\mathcal{D}$ which preserves the symplectic structures ϵ_+ and ϵ_-. Then the integrability condition for $[\xi]$ can be expressed as (see [7,19]):

$$\epsilon_+(\mathcal{D}_Z \xi, \xi) = 0, \quad \forall Z \in T_\xi M. \tag{3}$$

Considering the positive Weyl tensor W^+ of (M, g) as a section of the vector bundle $\mathrm{Sym}_0(\Lambda_+ M)$ we denote by Ψ the corresponding (real) section of the complex bundle $\Sigma_+^4 M$. Then for any complex vector fields $X_k = \xi_k \otimes \eta_k$, $k = 1, 2, 3, 4$ we have

$$W^+(X_1, X_2, X_3, X_4) = \epsilon_-(\eta_1, \eta_2)\epsilon_-(\eta_3, \eta_4)\Psi(\xi_1, \xi_2, \xi_3, \xi_4). \tag{4}$$

We will say that a direction $[\xi]$ in $\Sigma_+ M$ (resp. a positive orthogonal almost-complex structure J_ξ) is *principal* if $\Psi(\xi, \xi, \xi, \xi) = 0$.

For a principal direction $[\xi]$ we define its *multiplicity* as the largest number $1 \le p \le 4$ such that

$$\Psi(\underbrace{\xi, \xi, \ldots, \xi}_{(5-p)\,\text{times}}, \zeta_1, \zeta_2, \ldots, \zeta_{p-1}) \equiv 0, \quad \forall \zeta_k \in \Sigma_+ M, \quad k = 1, \ldots, p-1. \tag{5}$$

Using (4), we obtain the following characterization of the principal directions in $\Sigma_+ M$ in terms of the corresponding almost-complex structures:

Lemma 1. *Let (M, g) be an oriented Riemannian four-manifold with nowhere vanishing positive Weyl tensor. Let $[\xi]$ be a section of $\mathbf{P}(\Sigma_+ M)$. Denote by J_ξ the corresponding positive orthogonal almost-complex structure and by $\Lambda_0^+ M$ the bundle of J_ξ-anti-invariant 2-forms. Then*

(i) *$[\xi]$ is a principal direction iff W^+ has no component in Hom $(\Lambda_0^+ M)$;*
(ii) *$[\xi]$ has multiplicity 2 iff the spectrum of W^+ is degenerate and F_ξ lies in the eigenspace of W^+ corresponding to the simple eigenvalue;*
(iii) *$[\xi]$ has multiplicity at most 2.*

Proof. Since any $(1, 0)$-vector has the form $\xi \otimes \eta$ for some $\eta \in \Sigma_- M$, it follows from (4) that ξ is principal if and only if W^+ has no component in $\Lambda_\xi^{0,2} M \otimes \Lambda_\xi^{0,2} M$ or equivalently in $\mathrm{Hom}(\Lambda_0^+ M)$. In this case W^+ has the following form (see Section 2.2 in [2]):

$$W^+ = a(\tfrac{3}{4} F_\xi \otimes F_\xi - \tfrac{1}{2} Id) + \tfrac{1}{2}(\phi \otimes F_\xi + F_\xi \otimes \phi), \tag{6}$$

where a is a real function and ϕ is an element of $\Lambda_0^+ M$. Moreover, according to (4) ξ has multiplicity 2 iff $W^+(Z_1, Z_2, Z_3, X) = 0$ for any $(1, 0)$-vectors Z_1, Z_2, Z_3 and any vector X. Using the latter condition and (6) we prove (ii). If ξ has multiplicity 3, then we have that $W^+(Z_1, Z_2, X, Y) = 0$ for any $(1, 0)$-vectors Z_1, Z_2 and for any vectors X, Y. But this implies $a = 0, \phi = 0$, i.e. W^+ vanishes, a contradiction. □

Remarks

(1) It is well known (see for example [19,22]) that for an integrable complex structure J the positive Weyl tensor W^+ has no component in Hom $(\Lambda_0^+ M)$, so any integrable section of $\mathbf{P}(\Sigma_+ M)$ is principal.

(2) At a point where W^+ does not vanish the principal directions $[\xi]$ are determined by the eigenspaces of W^+ up to quaternionic conjugation as follows (see [20] and [2, Remark 1]):

(i) If W^+ is not degenerate at point $x \in M$, i.e. W^+ has three distinct eigenvalues $\lambda_+ > \lambda_0 > \lambda_-$ at x, then there are two (up to sign) different *principal* positive orthogonal almost-complex structures J' and J'' whose Kähler forms F' and F'' are respectively given by:

$$F' = \frac{(\lambda_+ - \lambda_0)^{1/2}}{(\lambda_+ - \lambda_-)^{1/2}} F_- - \frac{(\lambda_0 - \lambda_-)^{1/2}}{(\lambda_+ - \lambda_-)^{1/2}} F_+,$$

$$F'' = \frac{(\lambda_+ - \lambda_0)^{1/2}}{(\lambda_+ - \lambda_-)^{1/2}} F_- + \frac{(\lambda_0 - \lambda_-)^{1/2}}{(\lambda_+ - \lambda_-)^{1/2}} F_+,$$

where F_+ (resp. F_-) denotes one of the two generators of the (λ_+)-eigenspace (resp. (λ_-)-eigenspace) of W^+ with square-norm equal to 2.

(ii) If W^+ is degenerate, but non-zero at a point $x \in M$ there is one (determined up to sign) *principal* positive orthogonal almost-complex structure J, whose Kähler form F is the generator of the simple eigenspace of W^+ with square-norm equal to 2.

3. Null directions of oriented Lorentzian four-manifolds

In this section we will briefly recall some facts concerning the geometry of null directions of an oriented Lorentzian four-manifold (M, g). A more detailed discussion, proofs and references can be found in [7,14].

The group $SL(2, \mathbf{C})$ is the two-fold simply connected covering of the connected component $SO_0(1,3)$ of the group of the positive Lorentzian transformations. The (real) spinor representations Σ and $\overline{\Sigma}$ of $SL(2, \mathbf{C})$ are obtained by the action of $SL(2, \mathbf{C})$ on the spaces $\mathbf{C}^2$ and $\overline{\mathbf{C}}^2$ respectively. (Here $\overline{\mathbf{C}}^2$ denotes the complex conjugate of $\mathbf{C}^2$). The *complex* symplectic structures ϵ and $\overline{\epsilon}$ of Σ and $\overline{\Sigma}$ identify Σ and $\overline{\Sigma}$ with the corresponding dual spaces and we have the following decomposition:

$$\Sigma^p \otimes \Sigma^q = \Sigma^{p+q} \oplus \Sigma^{p+q-2} \oplus \cdots \oplus \Sigma^{p-q}, \quad p \geq q,$$

where Σ^k denotes the kth symmetric tensor product of Σ. (The same formula holds for $\overline{\Sigma}$).

These algebraic considerations have an immediate geometric interpretation when (M, g) is an oriented Lorentzian four-manifold with a spinorial structure, i.e. with a principal $SL(2, \mathbf{C})$-bundle $\overline{Q}$, which covers the principal $SO_0(1,3)$-bundle Q of the positive g-orthonormal frames of TM. Then we may identify the complexified tangent bundle $TM \otimes \mathbf{C}$ with $\Sigma M \otimes \overline{\Sigma} M$ where ΣM and $\overline{\Sigma} M$ denote the rank 2 (complex) vector bundles associated to the corresponding representations of $SL(2, \mathbf{C})$ via $\overline{Q}$. Then the $\mathbf{C}$-linear extension of the metric g satisfies

$$g(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) = \epsilon(\xi_1, \xi_2)\overline{\epsilon}(\eta_1, \eta_2), \quad \forall \xi_1, \xi_2 \in \Sigma M, \quad \forall \eta_1, \eta_2 \in \overline{\Sigma} M.$$

The Hodge star-operator $*$ of (M, g) induces a complex structure on $\Lambda^2 M$, so that we have the following identifications of (complex) vector bundles:

$$(\Lambda^2 M, *) \cong \Sigma^2 M, \quad \mathrm{Sym}_0(\Lambda^2 M, *) \cong \Sigma^4 M. \tag{7}$$

The Levi–Civita connection D of (M, g) induces a connection $\mathcal{D}$ on $\overline{Q}$. We will denote also by $\mathcal{D}$ the linear connection induced on any of the vector bundles ΣM and $\overline{\Sigma} M$ which preserve the symplectic structures ϵ and $\overline{\epsilon}$.

For any point $x \in M$ the set of *null directions* of (TM_x, g), i.e. the directions in TM_x generated by a non-zero g-isotropic vector l, can be identified with $\mathbf{P}(\Sigma M_x)$ since any such a vector l corresponds to an element of $\Sigma M_x \otimes \overline{\Sigma} M_x$ of the form $\xi \otimes \overline{\xi}$, where ξ is a non-vanishing element of ΣM_x and $\overline{\xi}$ denotes the complex conjugate of ξ. For any section $[\xi]$ of $\mathbf{P}(\Sigma M)$ denote by $T_\xi M$ the rank 2 complex subbundle of $TM \otimes \mathbf{C}$ of the elements of $TM \otimes \mathbf{C}$ which annihilate ξ, i.e. $T_\xi M = \{\xi \otimes \eta : \eta \in \overline{\Sigma} M\}$. Then a section $[\xi]$ of $\mathbf{P}(\Sigma M)$ is said to be *integrable* or a *shear-free congruence* if

$$\epsilon(\mathcal{D}_Z \xi, \xi) = 0, \quad \forall Z \in T_\xi M.$$

Note that the above condition is conformally invariant and can be also expressed by:

$$D_{Z_i} Z_j \in T_\xi M, \quad \forall Z_i, Z_j \in T_\xi M. \tag{8}$$

Considering the Weyl tensor W of (M, g) as a $\mathbf{C}$-linear symmetric traceless endomorphism of $(\Lambda^2 M, *)$, we denote by Ψ the corresponding (complex) section of $\Sigma^4 M$, via (7).

A null direction $[\xi]$ is called *principal* if it satisfies the property $\Psi(\xi, \xi, \xi, \xi) = 0$. (The principal null directions have first been defined by Cartan in [5] who considered them as preferable null directions, naturally determined by the geometry of (M, g)). It follows from the definition of Ψ that a null direction $[\xi]$ is principal iff $W(Z_1, Z_2, Z_3, Z_4) = 0$, $\forall Z_k \in T_\xi M, k = 1, \ldots, 4$. In particular, because of (8), one can see that any integrable null direction $[\xi]$ is principal. We also note that, at the points of M where W does not vanish, there is at most four different principal null directions.

The multiplicity of a principal null direction $[\xi]$ is determined by (5). By contrast with the Riemannian case, all possibilities for the multiplicity of a principal null direction can appear. The *Petrov classification* of the Weyl curvature tensors describes the situation (at a point of M) as follows:

(i) Class I: There are exactly four principal null directions (of multiplicity 1) of W, i.e. the spectrum of W (considered as a traceless $\mathbf{C}$-linear endomorphism of $(\Lambda^2 M, *)$) is not degenerate.
(ii) Class II: W has exactly 3 null directions, one of which has multiplicity 2 and the others have multiplicity 1.
(iii) Class D : W has exactly 2 null directions of multiplicity 2.

(iv) Class III: W has 2 null directions, one of which has multiplicity 3 and the other has multiplicity 1.
(v) Class N: W has one null direction of multiplicity 4.
(vi) Class O: W has infinitely many null directions, i.e. $W = 0$.

4. Proof of Theorems A, B and C

We will proceed with both the Riemannian and Lorentzian cases. Let (M, g) be either an oriented Riemannian or Lorentzian four-manifold. We denote by Ψ the positive Weyl tensor, considered as a section of $\Sigma_+^4 M$ in the Riemannian case and the Weyl tensor, considered as a section of $\Sigma^4 M$ when (M, g) is Lorentzian. The covariant derivative $\mathcal{D}\Psi$ of Ψ is a section of $\Sigma_+ M \otimes \Sigma_- M \otimes \Sigma_+^4 M = \Sigma_+^3 M \otimes \Sigma_- M \oplus \Sigma_+^5 M \otimes \Sigma_- M$ (resp. of $\Sigma^3 M \otimes \overline{\Sigma} M \oplus \Sigma^5 M \otimes \overline{\Sigma} M$). We denote by $\delta\Psi$ and $\mathcal{P}\Psi$ respectively the projections of $\mathcal{D}\Psi$ into $\Sigma_+^3 M \otimes \Sigma_- M$ and $\Sigma_+^5 M \otimes \Sigma_- M$ in the Riemannian case and into $\Sigma^3 M \otimes \overline{\Sigma} M$ and $\Sigma^5 M \otimes \overline{\Sigma} M$ in the Lorentzian case. Then we have

$$
\begin{aligned}
\mathcal{P}\Psi(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5; \eta) &= \frac{1}{8}\sum_{i=2}^{5}[(\mathcal{D}_{\xi_1\otimes\eta}\Psi)(\xi_i, \xi_{i+1}, \xi_{i+2}, \xi_{i+3}) \\
&\quad + (\mathcal{D}_{\xi_i\otimes\eta}\Psi)(\xi_1, \xi_{i+1}, \xi_{i+2}, \xi_{i+3})], \qquad (9) \\
(\mathcal{D}_{\zeta\otimes\eta}\Psi)(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{1}{8}\sum_{i=1}^{4}[(\mathcal{D}_{\zeta\otimes\eta}\Psi)(\xi_i, \xi_{i+1}, \xi_{i+2}, \xi_{i+3}) \\
&\quad - (\mathcal{D}_{\xi_i\otimes\eta}\Psi)(\zeta, \xi_{i+1}, \xi_{i+2}, \xi_{i+3})] \\
&\quad + \mathcal{P}\Psi(\zeta, \xi_1, \xi_2, \xi_3, \xi_4; \eta) \\
&= \frac{1}{4}[\sum_{i=1}^{4}\epsilon_+(\zeta, \xi_i)\delta\Psi(\xi_{i+1}, \xi_{i+2}, \xi_{i+3}; \eta)] \\
&\quad + \mathcal{P}\Psi(\zeta, \xi_1, \xi_2, \xi_3, \xi_4; \eta), \qquad (10)
\end{aligned}
$$

where ζ and ξ_i ($i = 1, \ldots, 5, \xi_{i+5} := \xi_i$) are arbitrary sections of $\Sigma_+ M$ (resp. of ΣM) and η is a section of $\Sigma_- M$ (resp. of $\overline{\Sigma} M$).

Note that in the Riemannian case $-\delta\Psi$ corresponds to the codifferential δW^+ of the positive Weyl tensor considered as a section of the rank 8 real vector bundle *Ker*[*trace* : $\Lambda^1 M \otimes \Lambda^+ M \mapsto \Lambda^1 M$] while $\mathcal{P}\Psi$ represents the complementary part PW^+ of δW^+ in the irreducible $SO(4)$-decomposition of the covariant derivative DW^+ of the positive Weyl tensor. More precisely, it follows from (4) that for any complex vector fields $X_k = \xi_k \otimes \eta_k, k = 1, 2, \ldots, 5$ we have:

$$(D_{X_1} W^+)(X_2, X_3, X_4, X_5) = \epsilon_-(\eta_2, \eta_3)\epsilon_-(\eta_4, \eta_5)(\mathcal{D}_{\xi_1\otimes\eta_1}\Psi)(\xi_2, \xi_3, \xi_4, \xi_5), \qquad (11)$$

$$PW^+(X_1, X_2, X_3, X_4, X_5) = \epsilon_-(\eta_2, \eta_3)\epsilon_-(\eta_4, \eta_5)\mathcal{P}\Psi(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5; \eta_1), \qquad (12)$$

$$-\delta W^+(X_1, X_2, X_3) = \epsilon_-(\eta_2, \eta_3)\delta\Psi(\xi_1, \xi_2, \xi_3; \eta_1). \qquad (13)$$

(The same formulae hold in the Lorentzian case if we consider the Weyl tensor instead of the positive Weyl tensor.)

Now, it follows easily from Lemma 1, (11), (12) and (13) that Theorem A and Theorem B can be rewritten in the following "spinorial" form (which is also valid in the Lorentzian case).

Theorem 1. *Let (M, g) be an oriented Riemannian (resp. Lorentzian) four-manifold and let ξ be a principal direction in Σ_+M (resp. ΣM). Then*

(i) *if ξ has multiplicity $p = 2$ (resp. $2 \leq p \leq 4$) then ξ is* integrable *iff*

$$\delta\Psi(\underbrace{\xi, \xi, \ldots, \xi}_{(5-p)\ \textit{times}}, , , \ldots, ; \eta) = 0, \quad \forall \eta \in \Sigma_-M (\text{resp. } \overline{\Sigma}M); \tag{14}$$

(ii) *if ξ is integrable and $\delta\Psi$ satisfies*

$$\delta\Psi(\xi, \xi, \xi; \eta) = 0, \ \forall \eta \in \Sigma_-M \ (\textit{resp. } \overline{\Sigma}M), \tag{15}$$

then ξ has multiplicity at least 2.

Theorem 2. *Let (M, g) be an oriented Riemannian (resp. Lorentzian) four-manifold and let ξ be a principal direction in Σ_+M (resp. ΣM). Then*

(i) *if ξ has multiplicity p, $p = 1, 2$ (resp. $1 \leq p \leq 4$), then ξ is* integrable *iff*

$$\mathcal{P}\Psi(\underbrace{\xi, \xi, \ldots, \xi}_{(6-p)\ \textit{times}}, , \ldots, ; \eta) = 0, \quad \forall \eta \in \Sigma_-M \ (\textit{resp. } \overline{\Sigma}M); \tag{16}$$

(ii) *if ξ integrable and $\mathcal{P}\Psi$ satisfies*

$$\begin{aligned} \mathcal{P}\Psi(\xi, \xi, \xi, \xi, \zeta; \eta) = 0, \quad & \forall \zeta \in \Sigma_+M \ (\text{resp. } \Sigma M), \\ & \forall \eta \in \Sigma_-M \ (\textit{resp. } \overline{\Sigma}M), \end{aligned} \tag{17}$$

then ξ has multiplicity at least 2.

Proof of Theorems 1 and 2

(i) Let $[\xi]$ be a principal direction in Σ_+M (resp. ΣM). If $[\xi]$ has multiplicity 1, then (16) becomes $\mathcal{P}\Psi(\xi, \xi, \xi, \xi, \xi; \eta) = 0$, which, according to (10), can be rewritten as $(\mathcal{D}_Z\Psi)(\xi, \xi, \xi, \xi) = 0, \forall Z \in T_\xi M$. When the multiplicity p of $[\xi]$ is at least 2, we claim that any of conditions (14) and (16) is equivalent to the following:

$$\begin{aligned} &(\mathcal{D}_Z\Psi)(\underbrace{\xi, \xi, \ldots, \xi}_{(5-p)\ \text{times}}\, \zeta_1, \zeta_2, \ldots, \zeta_{p-1}) = 0, \\ &\forall \zeta_k \in \Sigma_+M \ (\text{resp. } \Sigma M), \ \forall Z \in T_\xi M. \end{aligned} \tag{18}$$

Indeed, let $p = 2$ (similar arguments are valid in Lorentzian case for $p > 2$). Then, using (9) and the fact that $\Psi(\xi, \xi, \xi, \zeta) = 0, \ \forall \zeta \in \Sigma_+M$, we obtain

$$\begin{aligned} \mathcal{P}\Psi(\xi, \xi, \xi, \xi, \zeta; \eta) &= \tfrac{1}{2}((\mathcal{D}_{\xi\otimes\eta}\Psi)(\xi, \xi, \xi, \zeta) + (\mathcal{D}_{\zeta\otimes\eta}\Psi)(\xi, \xi, \xi, \xi)) \\ &= \tfrac{1}{2}(\mathcal{D}_Z\Psi)(\xi, \xi, \xi, \zeta), \end{aligned} \tag{19}$$

where $Z = \xi \otimes \eta$ belongs to $T_\xi M$.

From (10) we also get

$$\epsilon_+(\zeta,\xi)\delta\Psi(\xi,\xi,\xi;\eta) = -\mathcal{P}\Psi(\zeta,\xi,\xi,\xi,\xi;\eta),$$
$$\forall\zeta\in\Sigma_+M,\quad\forall\eta\in\Sigma_-M, \tag{20}$$

which together with (19) proves the claim.

Now, since ξ has multiplicity p, (18) gives that for any sections ζ_k, $k=1,\dots,p-1$, of Σ_+M

$$(\mathcal{D}_Z\Psi)(\underbrace{\xi,\xi,\dots,\xi}_{(5-p)\text{ times}},\zeta_1,\zeta_2,\dots,\zeta_{p-1})$$
$$= -(5-p)\Psi(\mathcal{D}_Z\xi,\ \underbrace{\xi,\dots,\xi}_{(4-p)\text{ times}}\ ,\zeta_1,\zeta_2,\dots,\zeta_{p-1}).$$

Thus condition (18) is satisfied iff $\epsilon_+(\mathcal{D}_Z\xi,\xi)=0$, i.e. iff ξ is integrable. This proves Theorem 1-(i) and Theorem 2-(i).

(ii) Suppose that $[\xi]$ is an integrable section of $\mathbf{P}(\Sigma_+M)$ (the arguments in the Lorentzian case are the same). Since ξ is principal we have $\Psi(\xi,\xi,\xi,\zeta)=\mu\epsilon_+(\xi,\zeta), \forall\zeta\in\Sigma_+M$. We have to prove that any of the condition (15) or (17) implies $\mu\equiv 0$. Suppose that $\mu\neq 0$. Multiplying ξ by an appropriate non-vanishing function we can assume that μ is a constant. Then, by (9) and (10), we obtain

$$\mathcal{P}\Psi(\xi,\xi,\xi,\xi,\zeta;\eta) = -\mu\epsilon_+(\mathcal{D}_Z\xi,\zeta) - 2\mu\epsilon_+(\mathcal{D}_{\zeta\otimes\eta}\xi,\xi); \tag{21}$$

$$\begin{aligned}\epsilon_+(\xi,\zeta)\delta\Psi(\xi,\xi,\xi;\eta) &= \mathcal{P}\Psi(\xi,\xi,\xi,\xi,\zeta;\eta) - (\mathcal{D}_{\zeta\otimes\eta}\Psi)(\xi,\xi,\xi,\xi)\\ &= -\mu\epsilon_+(\mathcal{D}_Z\xi,\zeta) + 2\mu\epsilon_+(\mathcal{D}_{\zeta\otimes\eta}\xi,\xi),\end{aligned} \tag{22}$$

where $Z=\xi\otimes\eta$ is an arbitrary element of $T_\xi M$.

Now suppose that either (15) or (17) holds on (M,g). Then formulas (21) and (22) give respectively:

$$\epsilon_+(\mathcal{D}_{\xi\otimes\eta}\xi,\zeta) + 2\epsilon_+(\mathcal{D}_{\zeta\otimes\eta}\xi,\xi) = 0,\quad \forall\zeta\in\Sigma_+M\ \ \forall\eta\in\Sigma_-M \tag{23}$$
$$\epsilon_+(\mathcal{D}_{\xi\otimes\eta}\xi,\zeta) - 2\epsilon_+(\mathcal{D}_{\zeta\otimes\eta}\xi,\xi) = 0,\quad \forall\zeta\in\Sigma_+M\ \ \forall\eta\in\Sigma_-M. \tag{24}$$

Put $Z_j=\xi\otimes\eta_j$, $X_j=\zeta\otimes\eta_j$; $j=1,2$. Since ξ is integrable, we get from (23) and (24) respectively:

$$\epsilon_+(\mathcal{D}^2_{Z_1,Z_2}\xi,\zeta) = -2\epsilon_+(\mathcal{D}^2_{Z_1,X_2}\xi,\xi) \tag{25}$$

$$\epsilon_+(\mathcal{D}^2_{Z_1,Z_2}\xi,\zeta) = 2\epsilon_+(\mathcal{D}^2_{Z_1,X_2}\xi,\xi) \tag{26}$$

and hence we obtain that either

$$\epsilon_+(\mathcal{R}_{Z_1,Z_2}\xi,\zeta) = -2\epsilon_+(\mathcal{D}^2_{Z_1,X_2}\xi,\xi) + 2\epsilon_+(\mathcal{D}^2_{Z_2,X_1}\xi,\xi) \tag{27}$$

or

$$\epsilon_+(\mathcal{R}_{Z_1,Z_2}\xi,\zeta) = 2\epsilon_+(\mathcal{D}^2_{Z_1,X_2}\xi,\xi) - 2\epsilon_+(\mathcal{D}^2_{Z_2,X_1}\xi,\xi), \tag{28}$$

where $\mathcal{R}$ denotes the curvature operator acting on the sections of $\Sigma_+ M$. Also by using (23) and (24) one can rewrite the right hand sides of (27) and (28) respectively as:

$$-2\epsilon_+(\mathcal{R}_{Z_1,X_2}\xi,\xi) - 2\epsilon_+(\mathcal{R}_{X_1,Z_2}\xi,\xi)$$

and

$$2\epsilon_+(\mathcal{R}_{Z_1,X_2}\xi,\xi) + 2\epsilon_+(\mathcal{R}_{X_1,Z_2}\xi,\xi).$$

Finally, taking into account that

$$\epsilon_+(\mathcal{R}_{Z_1,Z_2}\xi,\zeta) = \epsilon_-(\eta_1,\eta_2)\Psi(\xi,\xi,\xi,\zeta)$$

and

$$\epsilon_+(\mathcal{R}_{Z_1,X_2}\xi,\xi) + \epsilon_+(\mathcal{R}_{X_1,Z_2}\xi,\xi) = 2\epsilon_-(\eta_1,\eta_2)\Psi(\xi,\xi,\xi,\zeta)$$

we obtain that $\Psi(\xi,\xi,\xi,\zeta) = 0$ in both cases, a contradiction. □

Proof of Theorem C. Suppose that $PW = 0$ and ξ is a principal null direction of W of multiplicity 1. According to Theorem 2-(i), ξ is integrable which contradicts Theorem 2-(ii). □

Remark. Besides the principal null directions of the Weyl curvature tensor of an oriented Lorentzian four-manifold (M, g), one can define principal null directions of the traceless Ricci tensor in the following way: consider the traceless Ricci tensor as a (real) section, say Φ, of the bundle $\Sigma^2 M \otimes \overline{\Sigma}^2 M$. A non-vanishing section ξ of ΣM is said to be *principal with respect to* Φ iff $\Phi(\xi,\xi,\eta_1,\eta_2) = 0$, $\forall \eta_1,\eta_2 \in \overline{\Sigma} M$. While principal null directions of the Weyl tensor always exist, the existence of principal null directions of the traceless Ricci tensor imposes constraints on the eigenvalues of the Ricci tensor, see [14, Chapter 8, Section 8]. An immediate consequence of the second Bianchi identity and Theorem 1-(ii) is the Robinson–Shild theorem [17] which says that any integrable null direction which is principal with respect to the Ricci tensor has multiplicity at least 2. Indeed, if ξ is a principal null direction of Φ then the Ricci tensor Ric satisfies $Ric(Z_i, Z_j) = 0$, $\forall Z_i, Z_j \in T_\xi M$. If moreover ξ is integrable, the latter equality implies that $(D_{Z_i} Ric)(Z_j, Z_k) = 0$, $\forall Z_i, Z_j, Z_k \in T_\xi M$ and hence, using the second Bianchi identity, we obtain $\delta W(Z_i, Z_j, Z_k) = 0$, $\forall Z_i, Z_j, Z_k \in T_\xi M$. Now, according to the Lorentzian version of (13), the above equality can be rewritten as $\delta\Psi(\xi,\xi,\xi;\eta) = 0$, $\forall \eta \in \overline{\Sigma} M$ and using Theorem 1-(ii) we infer that ξ has multiplicity at least 2.

In the Riemannian case a principal direction of the Ricci tensor Ric corresponds to a positive orthogonal almost-complex structure J for which Ric is J-invariant. If Ric is not a multiple of the metric such a structure exists whenever Ric has exactly two different eigenvalues of multiplicity 2. In such a case this structure is uniquely determined (up to sign) and it is *integrable* iff (14) holds [2, Proposition 4]. In particular, applying Theorem A in the case of a Hermitian surface with J-invariant Ricci tensor, we obtain the Riemannian analogue of the Robinson–Shild theorem mentioned in Section 1 [2, Theorem 2].

5. Isotropic 2-planes and orthogonal almost-complex structures of oriented pseudo-Riemannian four-manifolds with metrics of zero signature

Through this section we will suppose that (M, g) is an oriented pseudo-Riemannian four-manifold whose metric g is of signature 0.

The connected component $SO_0(2,2)$ of the structure group $SO(2,2)$ of (M, g) has the splitting $SO_0(2,2) = SL(2) \times SL(2)$ which allows one to define two real vector bundles (of rank 2) S_+M and S_-M associated with the action of $SL(2)$ on $\mathbf{R}^2$. We will call $S_\pm M$ positive and negative *real* spinor bundles of (M, g), respectively. Note that S_+M and S_-M can be equipped with canonical symplectic structures ϵ_+ and ϵ_- which identify $S_\pm M$ with the dual vector bundles and give rise to splitting (1). Moreover, the tangent bundle TM is isomorphic to $S_+M \otimes S_-M$ and identity (2) is satisfied for the metric g. Thus, we have the splitting

$$\Lambda^2 M = S_+^2 M \oplus S_-^2 M$$

of the bundle $\Lambda^2 M$ of 2-forms on M into the direct sum of the subbundles $\Lambda^+ M = S_+^2 M$ and $\Lambda^- M = S_-^2 M$ of self-dual and anti-self-dual 2-forms on M. (In this case the Hodge star-operator $*$ acts as an involutive endomorphism of $\Lambda^2 M$ and $\Lambda^\pm M$ are the (± 1)-eigenspaces of $*$).

For any point $x \in M$ real isotropic 2-planes at x can be identified with the fiber of the real projective bundles $\mathbf{P}(S_+M)$ and $\mathbf{P}(S_-M)$ at x (see [1]). Denoting by $\mathcal{D}$ the Levi-Civita connection on the vector bundle S_+M (resp. S_-M) we will consider *integrability* condition (3) for a section of $\mathbf{P}(S_+M)$ (resp. of $\mathbf{P}(S_-M)$), i.e. a section $[\xi]$ of $\mathbf{P}(S_+M)$ is said to be *integrable* if $\epsilon_+(\mathcal{D}_Z\xi, \xi) = 0$, $\forall Z \in T_\xi M = \{\xi \otimes \eta;\ \eta \in S_-M\}$, or equivalently, if $D_{Z_i} Z_j \in T_\xi M$, $\forall Z_i, Z_j \in T_\xi M$. Since the real 2-plane distribution $T_\xi M$ is isotropic the latter condition is equivalent to the usual integrability of $T_\xi M$.

The Weyl curvature tensor W can be considered as a section of the vector bundle $\mathrm{Sym}_0(\Lambda^2 M) = S_+^4 M \oplus S_-^4 M$ of traceless symmetric endomorphisms of $\Lambda^2 M$. If we denote by W^+ the positive Weyl tensor (i.e. the projection of W on $\mathrm{Sym}_0(\Lambda^+ M) = S_+^4 M$) and by Ψ the corresponding section of $S_+^4 M$, then all definitions concerning principal directions of Ψ given in Section 2 can be adapted to sections of $\mathbf{P}(S_+M)$ (see [1, Section 6]). In contrast to the Riemannian and Lorentzian cases where principal directions of the Weyl tensor always exist, the existence of principal directions of S_+M is related to the existence of real roots of a fourth-degree polynomial with real coefficients (see [1, Section 6]). To avoid this problem we consider the *complex* spinor bundles $\Sigma_\pm M = S_\pm \otimes \mathbf{C}$. Now, the complexified tangent bundle $TM \otimes \mathbf{C}$ is isomorphic to $\Sigma_+ M \otimes \Sigma_- M$ and the complex continuations of g, ϵ_+ and ϵ_- on $TM \otimes \mathbf{C}$ satisfy (2). Besides real isotropic 2-planes the complex projective bundles $\mathbf{P}(\Sigma_\pm M)$ parametrize g-orthogonal almost-complex structures of (M, g) by means of the following correspondence: Fix a point $x \in M$ and denote by $\tilde{\mathbf{P}}(S_+M_x)$ the image of $\mathbf{P}(S_+M_x)$ in $\mathbf{P}(\Sigma_+M_x)$ by the natural embedding of S_+M_x into Σ_+M_x. Then for any $[\xi] \in \mathbf{P}(\Sigma_+M_x) \backslash \tilde{\mathbf{P}}(S_+M_x)$ we consider the complex isotropic 2-plane $T_\xi M_x = \{\xi \otimes \eta;\ \eta \in \Sigma_- M_x\}$. The special choice of $[\xi]$ gives $TM_x \otimes \mathbf{C} = T_\xi M_x \oplus T_{\bar{\xi}} M_x$, where $\overline{\xi}$ denotes the complex conjugate of ξ. Hence the latter splitting defines a *negative* g-orthogonal almost-complex structure J_ξ on TM_x whose $(1,0)$-space is $T_\xi M_x$. Considerations in Section 2 concerning positive orthogonal almost-complex structures of an oriented Riemannian four-manifold can be repeated for negative orthogonal almost-complex structures of an oriented pseudo-Riemannian four-manifold with a metric of signature 0. In particular, for a section $[\xi]$ of $\mathbf{P}(\Sigma_+ M)\backslash\tilde{\mathbf{P}}(S_+M)$ integrability condition (3) means that J_ξ is an integrable almost-complex structure. Moreover, as in the Riemannian case, Ψ is a real section of $\Sigma^4_+ M$ and hence for a principal direction $[\xi] \in \mathbf{P}(\Sigma_+ M_x)\backslash\tilde{\mathbf{P}}(S_+ M_x)$ Lemma 1 holds. This shows that for negative principal orthogonal almost-complex structures of (M, g) Theorems 1 and 2 can be reformulated as Theorems A and B.

Proof of Theorem D. We first note that any smooth section F of $\Lambda^+ M$ such that $g(F, F) = 2$ is the Kähler form of a negative g-orthogonal almost-complex structure J. Replacing M by a two-fold covering, if necessary, the positive Weyl tensor W^+ can be written as: $W^+ = \frac{3}{4}\lambda F \otimes F - \frac{1}{2}\lambda Id$, where λ is a nowhere vanishing function on M (equal at each point to the simple eigenvalue of W^+) and F is a globally defined self-dual 2-form which generates the λ-eigenspace of W^+ at each point and satisfies $g(F, F) = 2$. Since the metric g is Einstein, the second Bianchi identity gives $\delta W^+ = 0$, and it follows from Theorem A that the negative orthogonal almost-complex structure J corresponding to F is integrable. Denote by θ the Lee form of (M, g, J) defined (as in the Riemannian case) by $\mathrm{d}F = \theta \wedge F$ or equivalently by $\theta = -\delta F \circ J$. From the above expression for W^+ we infer easily that $\delta W^+ \equiv 0$ iff $\theta + \frac{2}{3}\,\mathrm{d}\ln\lambda \equiv 0$ iff the metric $\overline{g} = \lambda^{2/3} g$ is Kählerian. Suppose that M is compact. If (M, J, g) is Kähler then the scalar curvature $\frac{1}{6}\lambda$ of M is a non-zero constant since W^+ does not identically vanish, so (M, J) is either a minimal ruled surface of geneus ≥ 2 or a minimal surface of class VII_0 with no global spherical shell, and with second Betti number even and positive according to [15, Corollary 1]. Let (M, J, g) be non-Kähler. Since both metrics $\overline{g}$ and g have J-invariant Ricci tensor, as in the Riemannian case (see [2, Theorem 2]) we obtain that $X = J grad_{\overline{g}}\lambda$ is a non-trivial (real) holomorphic vector field with zeroes, hence the Kodaira dimension of (M, J) is $-\infty$, see [4]. Now, according to [15, Theorem 4], (M, J) is either a ruled surface or a minimal surface of class VII_0 with no global spherical shell, and with second Betti number even and positive. □

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